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Teaching And Learning Hyperbolic Functions (V); Two Other Groups Of Properties Of Hyperbolic Functions

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Abstract: *In four recent papers with the same generic name as this one and numbered with (I), (II), (III), respectively (IV), I presented the definitions, the consequences immediate resulting from these and a series of 38 properties of hyperbolic functions, properties that we divided into four groups, as follows: A) "Trigonometric" properties - nine properties; B) The derivatives of hyperbolic functions - six properties; C) The primitives (indefinite integrals) of hyperbolic functions – six properties and D) The monotony and the invertibility of hyperbolic functions - 17 properties. That in paper (I). In paper (II) I continued this approach and I presented another 54 properties of these functions, properties that have divided into three groups, as follows: E) Other properties "trigonometric" - 42 properties; F) Immediate properties of the inverse of hyperbolic functions – six properties and G) The derivatives of the inverse of hyperbolic functions - six properties. In paper (III) also I continued this approach and I presented another 36 properties of these functions, properties that we will divide into three groups, as follows: H) Properties „integral" and rewithrrence formulas - 11 properties; I) Relations between the inverse of hyperbolic functions - five properties and J) Relations between the hyperbolic functions and the inverses of other hyperbolic functions - 20 properties. In paper (IV) continuing those presented in the first three papers I presented and proved another 32 properties of these functions, properties that properties that we will classify them in the same group K): Sums and differences of inverse hyperbolic functions. In this paper we will conclude the presentation of the properties of these functions with two more categories: L) - Other relationships between the inverses of hyperbolic functions (four properties) and M) - Taylor series expansion of hyperbolic functions and their inverses (12 properties).*

Key words: *hyperbolic sine, hyperbolic cosine, hyperbolic tangent, hyperbolic cotangent, hyperbolic secant, hyperbolic cosecant.*

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I. Introduction

As part of a larger project entitled "*Training and developing the competences of children, students and teachers to solve problems / exercises in Mathematics*", in four recent papers with the same generic name as this one and numbered with (I), (II), (III), respectively (IV), I presented the definitions, the consequences immediate resulting from these and a series of 38 properties of hyperbolic functions, properties that we divided into four groups, as follows: A) "*Trigonometric*" properties - nine properties; B) The derivatives of hyperbolic functions - six properties; C) The primitives (indefinite integrals) of hyperbolic functions – six properties and D) The monotony and the invertibility of hyperbolic functions - 17 properties. That in paper (I). In paper (II) I continued this approach and I presented another 54 properties of these functions, properties that have divided into three groups, as follows: E) Other properties "*trigonometric*" - 42 properties; F) Immediate properties of the inverse of hyperbolic functions – six properties and G) The derivatives of the inverse of hyperbolic functions - six properties. In paper (III) also I continued this approach and I presented another 36 properties of these functions, properties that we will divide into three groups, as follows: H) Properties „*integral*" and rewithrrence formulas - 11 properties; I) Relations between the inverse of hyperbolic functions - five properties and J) Relations between the hyperbolic functions and the inverses of other hyperbolic functions - 20 properties. In paper (IV) continuing those presented in the first three papers I presented and proved another 32 properties of these functions, properties that properties that we will classify them in the same group K): Sums and differences of inverse hyperbolic functions. In this paper we will conclude the presentation of the properties of these functions with two more categories: L) - Other relationships between the inverses of hyperbolic functions (four properties) and M) - Taylor series expansion of hyperbolic functions and their inverses (12 properties).

That is why I ask the reader attentive and interested in these matters to consider this paper as a continuation of the four papers mentioned above. In this regard we will keep the numbering of the results, thus, the results numbered with (2.1) to (2.16), respectively (3.1) to (3.30) are from paper (I) - i.e. (Vălcan, 2016), those numbered with (4.1) to (4.54′) are from paper (II) – i.e. (Vălcan, 2019), those numbered with (5.1) to (5.40′) are from paper (III) – i.e. (Vălcan, (1), 2020). Finally, results numbered with (6.1) to (6.38) are those presented and proven in paper (IV) – i.e. (Vălcan, (2), 2020).

As I mentioned in previous papers, these properties, as well as others that we will present and prove later, will be used in various applications in Algebra or Mathematical Analysis.

II. The main results

We present and prove here the 16 properties of the hyperbolic functions, mentioned above.

Proposition: *The following statements hold*:

L. Other relationships between the inverses of hyperbolic functions

1) *For every* $x\in\mathbf{R}$,

$$2\cdot sh^{-1}x=ch^{-1}(2\cdot x^2+1). \tag{7.1}$$

2) *For every* $x\in[1,+\infty)$,

$$2\cdot ch^{-1}x=ch^{-1}(2\cdot x^2-1). \tag{7.2}$$

3) *For every* $x\in\mathbf{R}$,

$$4\cdot sh^{-1}x=ch^{-1}(8\cdot x^4+8\cdot x^2+1). \tag{7.3}$$

4) *For every* $x\in[1,+\infty)$,

$$4\cdot ch^{-1}x=ch^{-1}(8\cdot x^4-8\cdot x^2+1). \tag{7.4}$$

M. Taylor series expansion of hyperbolic functions and their inverses

5) *For every* $x\in\mathbf{R}$,

$$shx=x+\frac{1}{3!}\cdot x^3+\frac{1}{5!}\cdot x^5+\frac{1}{7!}\cdot x^7+\Lambda+\frac{1}{(2\cdot n+1)!}\cdot x^{2\cdot n+1}+\Lambda$$

$$=\sum_{n=0}^{\infty}\frac{1}{(2\cdot n+1)!}\cdot x^{2\cdot n+1}. \tag{7.5}$$

6) *For every* $x\in\mathbf{R}$,

$$chx=1+\frac{1}{2!}\cdot x^2+\frac{1}{4!}\cdot x^4+\frac{1}{6!}\cdot x^6+\Lambda+\frac{1}{(2\cdot n)!}\cdot x^{2\cdot n}+\Lambda$$

$$=\sum_{n=0}^{\infty}\frac{1}{(2\cdot n)!}\cdot x^{2\cdot n}. \tag{7.6}$$

7) *For every* $x\in\mathbf{R}$,

$$schx=1-\frac{1}{2}\cdot x^2+\frac{5}{24}\cdot x^4-\frac{61}{720}\cdot x^6+\frac{277}{8064}\cdot x^8+\Lambda+\frac{E_{2\cdot n}}{(2\cdot n)!}\cdot x^{2\cdot n}+\Lambda$$

$$=\sum_{n=0}^{\infty}\frac{E_{2\cdot n}}{(2\cdot n)!}\cdot x^{2\cdot n}; \tag{7.7}$$

where $(E_k)_{k\geq 1}$ *is called the sequence of Euler numbers*.

8) *For every* $x\in\mathbf{R}^*$,

$$cshx=\frac{1}{x}-\frac{1}{6}\cdot x-\frac{7}{360}\cdot x^3-\frac{31}{15120}\cdot x^5-\frac{127}{604800}\cdot x^7-\Lambda+(-1)^n\cdot\frac{2\cdot(2^{2\cdot n-1}-1)\cdot B_{2\cdot n}}{(2\cdot n)!}\cdot x^{2\cdot n-1}+\Lambda$$

$$=\frac{1}{x}+\sum_{n=1}^{\infty}(-1)^n\cdot\frac{2\cdot(2^{2\cdot n-1}-1)\cdot B_{2\cdot n}}{(2\cdot n)!}\cdot x^{2\cdot n-1}; \tag{7.8}$$

where $(B_k)_{k\geq 1}$ *is called the sequence of Bernoulli numbers*.

9) *For every* $x\in(-1,1)$,

$$thx=x-\frac{1}{3}\cdot x^3+\frac{2}{15}\cdot x^5-\frac{17}{315}\cdot x^7+\frac{62}{2835}\cdot x^9+\Lambda+\frac{2^{2\cdot n}\cdot(2^{2\cdot n}-1)\cdot B_{2\cdot n}}{(2\cdot n)!}\cdot x^{2\cdot n-1}+\Lambda$$

$$=\sum_{n=1}^{\infty}\frac{2^{2\cdot n}\cdot(2^{2\cdot n}-1)\cdot B_{2\cdot n}}{(2\cdot n)!}\cdot x^{2\cdot n-1}; \tag{7.9}$$

where $(B_k)_{k\geq 1}$ *is the sequence of Bernoulli numbers*.

10) *For every* $x\in\mathbf{R}^*$,

$$cthx=\frac{1}{x}+\frac{1}{3}\cdot x-\frac{1}{45}\cdot x^3+\frac{2}{945}\cdot x^5-\frac{1}{4725}+\Lambda+\frac{2^{2\cdot n}\cdot B_{2\cdot n}}{(2\cdot n)!}\cdot x^{2\cdot n-1}+\Lambda$$

$$=\frac{1}{x}+\sum_{n=1}^{\infty}\frac{2^{2\cdot n}\cdot B_{2\cdot n}}{(2\cdot n)!}\cdot x^{2\cdot n-1}; \tag{7.10}$$

where $(B_k)_{k\geq 1}$ *is the sequence of Bernoulli numbers.*

11) *For every* $x\in(-1,1)$,

$$sh^{-1}x=x-\left(\frac{1}{2}\right)\cdot\frac{x^3}{3}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{x^5}{5}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{x^7}{7}+\Lambda+(-1)^n\cdot\frac{(2\cdot n)!\cdot}{2^{2\cdot n}\cdot(n!)^2}\cdot\frac{x^{2\cdot n+1}}{2\cdot n+1}+\Lambda$$

$$=\sum_{n=0}^{\infty}\left(\frac{(-1)^n\cdot(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\right)\cdot\frac{x^{2\cdot n+1}}{2\cdot n+1}; \tag{7.11}$$

or, for every $x\in(1,+\infty)$,

$$sh^{-1}x=ln(2\cdot x)+\left(\frac{1}{2}\right)\cdot\frac{1}{2\cdot x^2}-\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{4\cdot x^4}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{1}{6\cdot x^6}-\Lambda+(-1)^{n-1}\cdot\frac{(2\cdot n-1)!!\cdot}{2\cdot n\cdot(2\cdot n)!!}\cdot\frac{1}{x^{2\cdot n}}+\Lambda$$

$$=ln(2\cdot x)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot(2\cdot n-1)!!}{2\cdot n\cdot(2\cdot n)!!}\cdot\frac{1}{x^{2\cdot n}}; \tag{7.12}$$

or, for every $x\in(-\infty,-1)$,

$$sh^{-1}x=-ln(2\cdot|x|)-\left(\frac{1}{2}\right)\cdot\frac{1}{2\cdot x^2}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{4\cdot x^4}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{1}{6\cdot x^6}+\Lambda+(-1)^{n-1}\cdot\frac{(2\cdot n-1)!!\cdot}{2\cdot n\cdot(2\cdot n)!!}\cdot\frac{1}{x^{2\cdot n}}+\Lambda$$

$$=-ln(2\cdot|x|)-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot(2\cdot n-1)!!}{2\cdot n\cdot(2\cdot n)!!}\cdot\frac{1}{x^{2\cdot n}}. \tag{7.13}$$

12) *For every* $x\in(1,+\infty)$,

$$ch_1^{-1}x=-ln(2\cdot x)-\left[\left(\frac{1}{2}\right)\cdot\frac{1}{2\cdot x^2}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{4\cdot x^4}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{1}{6\cdot x^6}+\Lambda+\frac{(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\cdot\frac{1}{(2\cdot n)\cdot x^{2\cdot n}}+\Lambda\right]$$

$$=-ln(2\cdot x)-\sum_{n=1}^{\infty}\frac{(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\cdot\frac{1}{(2\cdot n)\cdot x^{2\cdot n}} \tag{7.14}$$

and

$$ch_2^{-1}x=ln(2\cdot x)+\left[\left(\frac{1}{2}\right)\cdot\frac{1}{2\cdot x^2}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{4\cdot x^4}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{1}{6\cdot x^6}+\Lambda+\frac{(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\cdot\frac{1}{(2\cdot n)\cdot x^{2\cdot n}}+\Lambda\right]$$

$$=ln(2\cdot x)+\sum_{n=1}^{\infty}\left(\frac{(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\right)\cdot\frac{1}{(2\cdot n)\cdot x^{2\cdot n}}. \tag{7.15}$$

13) *For every* $x\in(-1,1)$,

$$th^{-1}x=x+\frac{1}{3}\cdot x^3+\frac{1}{5}\cdot x^5+\frac{1}{7}\cdot x^7+\Lambda+\frac{x^{2\cdot n+1}}{2\cdot n+1}+\Lambda$$

$$=\sum_{n=0}^{\infty}\frac{x^{2\cdot n+1}}{2\cdot n+1}. \tag{7.16}$$

14) *For every* $x\in(-\infty,-1)\cup(1,+\infty)$,

$$cth^{-1}x=\frac{1}{x}+\frac{1}{3\cdot x^3}+\frac{1}{5\cdot x^5}+\frac{1}{7\cdot x^7}+\Lambda+\frac{1}{(2\cdot n+1)\cdot x^{2\cdot n+1}}+\Lambda$$

$$=\sum_{n=0}^{\infty}\frac{1}{(2\cdot n+1)\cdot x^{2\cdot n+1}}. \tag{7.17}$$

15) *For every* $x\in(0,1)$,

$$sch_1^{-1}x=-ln\left(\frac{2}{x}\right)-\left[\left(\frac{1}{2}\right)\cdot\frac{x^2}{2}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{x^4}{4}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{x^6}{6}+\Lambda+\frac{(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\cdot\frac{x^{2\cdot n}}{(2\cdot n)}+\Lambda\right]$$

$$=-ln\left(\frac{2}{x}\right)-\sum_{n=1}^{\infty}\frac{(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\cdot\frac{x^{2\cdot n}}{(2\cdot n)} \quad (7.18)$$

and

$$sch_2^{-1}x=ln\left(\frac{2}{x}\right)+\left[\left(\frac{1}{2}\right)\cdot\frac{x^2}{2}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{x^4}{4}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{x^6}{6}+\Lambda+\frac{(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\cdot\frac{x^{2\cdot n}}{(2\cdot n)}+\Lambda\right]$$

$$=ln\left(\frac{2}{x}\right)+\sum_{n=1}^{\infty}\frac{(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\cdot\frac{x^{2\cdot n}}{(2\cdot n)}. \quad (7.19)$$

***16)** The following statements hold*:

***a)** For every $x\in(-\infty,-1)$,*

$$csh_1^{-1}x=\frac{1}{x}-\left(\frac{1}{2}\right)\cdot\frac{1}{3\cdot x^3}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{5\cdot x^5}-$$

$$-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{1}{7\cdot x^7}+\Lambda\left(\frac{(-1)^n\cdot(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\right)\cdot\frac{1}{(2\cdot n+1)\cdot x^{2\cdot n+1}}+\Lambda$$

$$=\sum_{n=0}^{\infty}\left(\frac{(-1)^n\cdot(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\right)\cdot\frac{1}{(2\cdot n+1)\cdot x^{2\cdot n+1}}. \quad (7.20)$$

***b)** For every $x\in(-1,0)$,*

$$csh_1^{-1}x=-ln\left(\frac{2}{|x|}\right)-\left(\frac{1}{2}\right)\cdot\frac{x^2}{2}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{x^4}{4}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{x^6}{6}+\Lambda+\frac{(-1)^{n-1}\cdot(2\cdot n-1)!!}{(2\cdot n)!!}\cdot\frac{x^{2\cdot n}}{2\cdot n}+\Lambda$$

$$=-ln\left(\frac{2}{|x|}\right)-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot(2\cdot n-1)!!}{(2\cdot n)!!}\cdot\frac{x^{2\cdot n}}{2\cdot n}. \quad (7.21)$$

***c)** For every $x\in(0,1)$,*

$$csh_2^{-1}x=ln\left(\frac{2}{x}\right)+\left(\frac{1}{2}\right)\cdot\frac{x^2}{2}-\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{x^4}{4}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{x^6}{6}-\Lambda+\frac{(-1)^{n-1}\cdot(2\cdot n-1)!!}{(2\cdot n)!!}\cdot\frac{x^{2\cdot n}}{2\cdot n}+\Lambda$$

$$=ln\left(\frac{2}{x}\right)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot(2\cdot n-1)!!}{(2\cdot n)!!}\cdot\frac{x^{2\cdot n}}{2\cdot n}. \quad (7.22)$$

***d)** For every $x\in(1,+\infty)$,*

$$csh_2^{-1}x=\frac{1}{x}-\left(\frac{1}{2}\right)\cdot\frac{1}{3\cdot x^3}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{5\cdot x^5}-$$

$$\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{1}{7\cdot x^7}+\Lambda+\left(\frac{(-1)^n\cdot(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\right)\cdot\frac{1}{(2\cdot n+1)\cdot x^{2\cdot n+1}}+\Lambda$$

$$=\sum_{n=0}^{\infty}\left(\frac{(-1)^n\cdot(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\right)\cdot\frac{1}{(2\cdot n+1)\cdot x^{2\cdot n+1}}. \quad (7.23)$$

***Proof*: 1)** As shown up here, for every $x\in\mathbf{R}$, the following equivalences hold:

$2\cdot sh^{-1}x=ch^{-1}(2\cdot x^2+1) \Leftrightarrow ch(2\cdot sh^{-1}x)=ch(ch^{-1}(2\cdot x^2+1))$

$\Leftrightarrow 2\cdot sh^2(sh^{-1}x)+1=ch(ch^{-1}(2\cdot x^2+1))$ (according to the equality (4.4″))

$\Leftrightarrow 2\cdot x^2+1=2\cdot x^2+1$;

which shows that the equality (7.1) holds.

***Otherwise*:** For every $x\in\mathbf{R}$, according to the equality (3.22), we have the equalities:

(1) $2\cdot sh^{-1}x=2\cdot ln(x+\sqrt{x^2+1})=ln(x+\sqrt{x^2+1})^2$

$=ln(x^2+2\cdot x\cdot\sqrt{x^2+1}+x^2+1)$

$=ln(2\cdot x^2+1+2\cdot x\cdot\sqrt{x^2+1})$,

and, according to the equalities (3.23), respective (3.24), we have the following equalities:

(2) $ch_1^{-1}(2\cdot x^2+1)=ln(2\cdot x^2+1-\sqrt{(2\cdot x^2+1)^2-1})=ln(2\cdot x^2+1-\sqrt{4\cdot x^4+4\cdot x^2})$

$=\ln(2\cdot x^2+1-2\cdot|x|\cdot\sqrt{x^2+1})$

$=\ln(2\cdot x^2+1+2\cdot x\cdot\sqrt{x^2+1})$, (because $x\le 0$)

respective:

(3) $ch_2^{-1}(2\cdot x^2+1)=\ln(2\cdot x^2+1+\sqrt{(2\cdot x^2+1)^2-1})=\ln(2\cdot x^2+1+\sqrt{4\cdot x^4+4\cdot x^2})$

$=\ln(2\cdot x^2+1+2\cdot|x|\cdot\sqrt{x^2+1})$

$=\ln(2\cdot x^2+1+2\cdot x\cdot\sqrt{x^2+1})$ (because $x\ge 0$).

From the equalities (1), (2) and (3) it follows that, for every $x\in\mathbf{R}$, the equality (7.1) holds.

2) As shown up here, for every $x\in\mathbf{R}$, the following equivalences hold:

$2\cdot ch^{-1}x=ch^{-1}(2\cdot x^2-1)\Leftrightarrow ch(2\cdot ch^{-1}x)=ch(ch^{-1}(2\cdot x^2-1))$

$\Leftrightarrow 2\cdot ch^2(ch^{-1}x)-1=ch(ch^{-1}(2\cdot x^2-1))$ (according to the equality (4.4′))

$\Leftrightarrow 2\cdot x^2-1=2\cdot x^2-1$;

which shows that the equality (7.2) holds.

***Otherwise*:** For every $x\in[1,+\infty)$, according to the equality (3.23), we have the equalities:

(1) $2\cdot ch_1^{-1}x=2\cdot\ln(x-\sqrt{x^2-1})$

$=\ln(x-\sqrt{x^2-1})^2$

$=\ln(x^2-2\cdot x\cdot\sqrt{x^2-1}+x^2-1)$

$=\ln(2\cdot x^2-1-2\cdot x\cdot\sqrt{x^2-1})$,

and:

(2) $ch_1^{-1}(2\cdot x^2-1)=\ln(2\cdot x^2-1-\sqrt{(2\cdot x^2-1)^2-1})$

$=\ln(2\cdot x^2-1-\sqrt{4\cdot x^4-4\cdot x^2})$

$=\ln(2\cdot x^2-1-2\cdot|x|\cdot\sqrt{x^2-1})$

$=\ln(2\cdot x^2-1-2\cdot x\cdot\sqrt{x^2-1})$ (because $x\ge 1$).

From the equalities (1) and (2) it follows that, for every $x\in[1,+\infty)$, the equality (7.2) holds for function ch_1^{-1}. On the other hand, for every $x\in[1,+\infty)$, according to the equality (3.24), we have the equalities:

(3) $2\cdot ch_2^{-1}x=2\cdot\ln(x+\sqrt{x^2-1})$

$=\ln(x+\sqrt{x^2-1})^2$

$=\ln(x^2+2\cdot x\cdot\sqrt{x^2-1}+x^2-1)$

$=\ln(2\cdot x^2-1+2\cdot x\cdot\sqrt{x^2-1})$,

and:

(4) $ch_2^{-1}(2\cdot x^2-1)=\ln(2\cdot x^2-1+\sqrt{(2\cdot x^2-1)^2-1})$

$=\ln(2\cdot x^2-1+\sqrt{4\cdot x^4-4\cdot x^2})$

$=\ln(2\cdot x^2-1+2\cdot|x|\cdot\sqrt{x^2-1})$

$=\ln(2\cdot x^2-1+2\cdot x\cdot\sqrt{x^2-1})$ (because $x\ge 1$).

From the equalities (3) and (4) it follows that, for every $x\in[1,+\infty)$, the equality (7.2) holds (also) for function ch_2^{-1}. In conclusion, the assertion from the statement is true.

3) As shown up here, for every $x\in\mathbf{R}$, the following equivalences hold:

$4\cdot sh^{-1}x=ch^{-1}(8\cdot x^4+8\cdot x^2+1)\Leftrightarrow ch(4\cdot sh^{-1}x)=ch(ch^{-1}(8\cdot x^4+8\cdot x^2+1))$

$\Leftrightarrow 2\cdot sh^2(2\cdot sh^{-1}x)+1=ch(ch^{-1}(8\cdot x^4+8\cdot x^2+1))$ (according to the equality (4.4″))

$\Leftrightarrow 2\cdot(2\cdot sh(sh^{-1}x)\cdot ch(sh^{-1}x))^2+1=8\cdot x^4+8\cdot x^2+1$; (according to the equality (4.2))

$\Leftrightarrow 2\cdot 4\cdot sh^2(sh^{-1}x)\cdot ch^2(sh^{-1}x))^2+1=8\cdot x^4+8\cdot x^2+1$

$\Leftrightarrow 8\cdot x^2\cdot(1+sh^2(sh^{-1}x))+1=8\cdot x^4+8\cdot x^2+1$ (according to the equality (3.1)

$\Leftrightarrow 8\cdot x^2\cdot(1+x^2)+1=8\cdot x^4+8\cdot x^2+1$

$\Leftrightarrow 8\cdot x^4+8\cdot x^2+1=8\cdot x^4+8\cdot x^2+1$;

which shows that the equality (7.3) holds.

***Otherwise*:** For every $x\in\mathbf{R}$, according to the equality (3.22), we have the equalities:

(1) $4\cdot sh^{-1}x=4\cdot\ln(x+\sqrt{x^2+1})=\ln(x+\sqrt{x^2+1})^4$

$=\ln(x^4+4\cdot x^3\cdot\sqrt{x^2+1}+6\cdot x^2\cdot(\sqrt{x^2+1})^2+4\cdot x\cdot(\sqrt{x^2+1})^3+(\sqrt{x^2+1})^4)$

$=\ln(x^4+4\cdot x^3\cdot\sqrt{x^2+1}+6\cdot x^2\cdot(x^2+1)+4\cdot x\cdot(x^2+1)\cdot(\sqrt{x^2+1})+(x^2+1)^2)$

$=\ln(x^4+4\cdot x^3\cdot\sqrt{x^2+1}+6\cdot x^4+6\cdot x^2+4\cdot(x^3+x)\cdot(\sqrt{x^2+1})+x^4+2\cdot x^2+1)$

$=\ln(8\cdot x^4+8\cdot x^2+1+(8\cdot x^3+4\cdot x)\cdot(\sqrt{x^2+1}))$,

and, according to the equalities (3.23), respective (3.24), we have the following equalities:

(2) $ch_1^{-1}(8\cdot x^4+8\cdot x^2+1)=\ln(8\cdot x^4+8\cdot x^2+1-\sqrt{(8\cdot x^4+8\cdot x^2+1)^2-1})$

$=\ln(8\cdot x^4+8\cdot x^2+1-\sqrt{(8\cdot x^4+8\cdot x^2+1-1)\cdot(8\cdot x^4+8\cdot x^2+1+1)})$

$=\ln(8\cdot x^4+8\cdot x^2+1-\sqrt{(8\cdot x^4+8\cdot x^2)\cdot(8\cdot x^4+8\cdot x^2+2)})$

$=\ln(8\cdot x^4+8\cdot x^2+1-\sqrt{8\cdot x^2\cdot(x^2+1)\cdot 2\cdot(4\cdot x^4+4\cdot x^2+1)})$

$=\ln(8\cdot x^4+8\cdot x^2+1-\sqrt{16\cdot x^2\cdot(x^2+1)\cdot(2\cdot x^2+1)^2})$

$=\ln(8\cdot x^4+8\cdot x^2+1-4\cdot|x|\cdot(2\cdot x^2+1)\cdot\sqrt{x^2+1})$

$=\ln(8\cdot x^4+8\cdot x^2+1+4\cdot x\cdot(2\cdot x^2+1)\cdot\sqrt{x^2+1})$ (because $x\le 0$)

$=\ln(8\cdot x^4+8\cdot x^2+1+(8\cdot x^3+4\cdot x)\cdot\sqrt{x^2+1})$

respective:

(3) $ch_2^{-1}(8\cdot x^4+8\cdot x^2+1)=\ln(8\cdot x^4+8\cdot x^2+1+\sqrt{(8\cdot x^4+8\cdot x^2+1)^2-1})$

$=\ln(8\cdot x^4+8\cdot x^2+1+\sqrt{(8\cdot x^4+8\cdot x^2+1-1)\cdot(8\cdot x^4+8\cdot x^2+1+1)})$

$=\ln(8\cdot x^4+8\cdot x^2+1+\sqrt{(8\cdot x^4+8\cdot x^2)\cdot(8\cdot x^4+8\cdot x^2+2)})$

$=\ln(8\cdot x^4+8\cdot x^2+1+\sqrt{8\cdot x^2\cdot(x^2+1)\cdot 2\cdot(4\cdot x^4+4\cdot x^2+1)})$

$=\ln(8\cdot x^4+8\cdot x^2+1+\sqrt{16\cdot x^2\cdot(x^2+1)\cdot(2\cdot x^2+1)^2})$

$=\ln(8\cdot x^4+8\cdot x^2+1+4\cdot|x|\cdot(2\cdot x^2+1)\cdot\sqrt{x^2+1})$

$=\ln(8\cdot x^4+8\cdot x^2+1+4\cdot x\cdot(2\cdot x^2+1)\cdot\sqrt{x^2+1})$ (because $x\ge 0$)

$=\ln(8\cdot x^4+8\cdot x^2+1+(8\cdot x^3+4\cdot x)\cdot\sqrt{x^2+1})$.

From the equalities (1), (2) and (3) it follows that, for every $x\in\mathbf{R}$, the equality (7.3) holds.

4) As shown up here, for every $x\in\mathbf{R}$, the following equivalences hold:

$4\cdot ch^{-1}x=ch^{-1}(8\cdot x^4-8\cdot x^2+1)\Leftrightarrow ch(4\cdot ch^{-1}x)=ch(ch^{-1}(8\cdot x^4-8\cdot x^2+1))$

$\Leftrightarrow 2\cdot ch^2(2\cdot ch^{-1}x)-1=ch(ch^{-1}(8\cdot x^4-8\cdot x^2+1))$ (according to the equality (4.4′))

$\Leftrightarrow 2\cdot(2\cdot ch^2(ch^{-1}x)-1)^2-1=8\cdot x^4-8\cdot x^2+1$

$\Leftrightarrow 2\cdot(2\cdot x^2-1)^2-1=8\cdot x^4-8\cdot x^2+1$

$\Leftrightarrow 2\cdot(4\cdot x^4-4\cdot x^2+1)-1=8\cdot x^4-8\cdot x^2+1$

$\Leftrightarrow 8\cdot x^4-8\cdot x^2+2-1=8\cdot x^4-8\cdot x^2+1$

$\Leftrightarrow 8\cdot x^4-8\cdot x^2+1=8\cdot x^4-8\cdot x^2+1$;

which shows that the equality (7.4) holds.

***Otherwise*:** For every $x\in[1,+\infty)$, according to the equality (3.23), we have the equalities:

(1) $4\cdot ch_1^{-1}x=4\cdot\ln(x-\sqrt{x^2-1})=\ln(x-\sqrt{x^2-1})^4$

$=\ln(x^4-4\cdot x^3\cdot\sqrt{x^2-1}+6\cdot x^2\cdot(\sqrt{x^2-1})^2-4\cdot x\cdot(\sqrt{x^2-1})^3+(\sqrt{x^2-1})^4)$

$=\ln(x^4-4\cdot x^3\cdot\sqrt{x^2-1}+6\cdot x^2\cdot(x^2-1)-4\cdot x\cdot(x^2-1)\cdot(\sqrt{x^2-1})+(x^2-1)^2)$

$=\ln(x^4-4\cdot x^3\cdot\sqrt{x^2-1}+6\cdot x^4-6\cdot x^2-4\cdot(x^3-x)\cdot(\sqrt{x^2-1})+x^4-2\cdot x^2+1)$

$=\ln(8\cdot x^4-8\cdot x^2+1-(8\cdot x^3-4\cdot x)\cdot(\sqrt{x^2-1}))$,

and:

(2) $ch_1^{-1}(8\cdot x^4-8\cdot x^2+1)=\ln(8\cdot x^4-8\cdot x^2+1-\sqrt{(8\cdot x^4-8\cdot x^2+1)^2-1})$

$=\ln(8\cdot x^4-8\cdot x^2+1-\sqrt{(8\cdot x^4-8\cdot x^2+1-1)\cdot(8\cdot x^4-8\cdot x^2+1+1)})$

$=\ln(8\cdot x^4-8\cdot x^2+1-\sqrt{(8\cdot x^4-8\cdot x^2)\cdot(8\cdot x^4-8\cdot x^2+2)})$

$=\ln(8\cdot x^4-8\cdot x^2+1-\sqrt{8\cdot x^2\cdot(x^2-1)\cdot 2\cdot(4\cdot x^4-4\cdot x^2+1)})$

$=\ln(8\cdot x^4-8\cdot x^2+1-\sqrt{16\cdot x^2\cdot(x^2-1)\cdot(2\cdot x^2-1)^2})$

$=\ln(8\cdot x^4-8\cdot x^2+1-4\cdot|x|\cdot(2\cdot x^2-1)\cdot\sqrt{x^2-1})$

$=\ln(8\cdot x^4-8\cdot x^2+1-4\cdot x\cdot(2\cdot x^2-1)\cdot\sqrt{x^2-1})$ (because $x\geq 1$)

$=\ln(8\cdot x^4-8\cdot x^2+1-(8\cdot x^3-4\cdot x)\cdot\sqrt{x^2-1})$.

From the equalities (1) and (2) it follows that, for every $x\in[1,+\infty)$, the equality (7.4) holds for function ch_1^{-1}. On the other hand, for every $x\in[1,+\infty)$, according to the equality (3.24), we have the equalities:

(3) $4\cdot ch_2^{-1}x=4\cdot\ln(x+\sqrt{x^2-1})=\ln(x+\sqrt{x^2-1})^4$

$=\ln(x^4+4\cdot x^3\cdot\sqrt{x^2-1}+6\cdot x^2\cdot(\sqrt{x^2-1})^2+4\cdot x\cdot(\sqrt{x^2-1})^3+(\sqrt{x^2-1})^4)$

$=\ln(x^4+4\cdot x^3\cdot\sqrt{x^2-1}+6\cdot x^2\cdot(x^2-1)+4\cdot x\cdot(x^2-1)\cdot(\sqrt{x^2-1})+(x^2-1)^2)$

$=\ln(x^4+4\cdot x^3\cdot\sqrt{x^2-1}+6\cdot x^4-6\cdot x^2+4\cdot(x^3-x)\cdot(\sqrt{x^2-1})+x^4-2\cdot x^2+1)$

$=\ln(8\cdot x^4-8\cdot x^2+1+(8\cdot x^3-4\cdot x)\cdot(\sqrt{x^2-1}))$,

and:

(4) $ch_2^{-1}(8\cdot x^4-8\cdot x^2+1)=\ln(8\cdot x^4-8\cdot x^2+1+\sqrt{(8\cdot x^4-8\cdot x^2+1)^2-1})$

$=\ln(8\cdot x^4-8\cdot x^2+1+\sqrt{(8\cdot x^4-8\cdot x^2+1-1)\cdot(8\cdot x^4-8\cdot x^2+1+1)})$

$=\ln(8\cdot x^4-8\cdot x^2+1+\sqrt{(8\cdot x^4-8\cdot x^2)\cdot(8\cdot x^4-8\cdot x^2+2)})$

$=\ln(8\cdot x^4-8\cdot x^2+1+\sqrt{8\cdot x^2\cdot(x^2-1)\cdot 2\cdot(4\cdot x^4-4\cdot x^2+1)})$

$=\ln(8\cdot x^4-8\cdot x^2+1+\sqrt{16\cdot x^2\cdot(x^2-1)\cdot(2\cdot x^2-1)^2})$

$=\ln(8\cdot x^4-8\cdot x^2+1+4\cdot|x|\cdot(2\cdot x^2-1)\cdot\sqrt{x^2-1})$

$=\ln(8\cdot x^4-8\cdot x^2+1+4\cdot x\cdot(2\cdot x^2-1)\cdot\sqrt{x^2-1})$ (because $x\geq 1$)

$=\ln(8\cdot x^4-8\cdot x^2+1+(8\cdot x^3-4\cdot x)\cdot\sqrt{x^2-1})$.

From the equalities (3) and (4) it follows that, for every $x\in[1,+\infty)$, the equality (7.4) holds (also) for function ch_2^{-1}. In conclusion, the assertion from the statement is true.

5) For every $x\in\mathbf{R}$,

(1) $e^x=1+\frac{1}{1!}\cdot x+\frac{1}{2!}\cdot x^2+\frac{1}{3!}\cdot x^3+\frac{1}{4!}\cdot x^4+\frac{1}{5!}\cdot x^5+\Lambda+\frac{1}{n!}\cdot x^n+\Lambda$

and:

(2) $e^{-x}=1-\frac{1}{1!}\cdot x+\frac{1}{2!}\cdot x^2-\frac{1}{3!}\cdot x^3+\frac{1}{4!}\cdot x^4-\frac{1}{5!}\cdot x^5+\Lambda+(-1)^n\cdot\frac{1}{n!}\cdot x^n+\Lambda$

Subtracting member with member, the equalities (1) and (2), obtain that, for every $x\in\mathbf{R}$,

$$e^x-e^{-x}=2\cdot\left(\frac{1}{1!}\cdot x+\frac{1}{3!}\cdot x^3+\frac{1}{5!}\cdot x^5+\Lambda+\frac{1}{(2\cdot n+1)!}\cdot x^{2\cdot n+1}+\Lambda\right);$$

so, according to the equality (2.1), the equality (7.5) holds.

6) For every $x\in\mathbf{R}$,

(1) $e^x=1+\frac{1}{1!}\cdot x+\frac{1}{2!}\cdot x^2+\frac{1}{3!}\cdot x^3+\frac{1}{4!}\cdot x^4+\frac{1}{5!}\cdot x^5+\Lambda+\frac{1}{n!}\cdot x^n+\Lambda$

and:

(2) $e^{-x}=1-\frac{1}{1!}\cdot x+\frac{1}{2!}\cdot x^2-\frac{1}{3!}\cdot x^3+\frac{1}{4!}\cdot x^4-\frac{1}{5!}\cdot x^5+\Lambda+(-1)^n\cdot\frac{1}{n!}\cdot x^n+\Lambda$

Additing member with member, the equalities (1) and (2), obtain that, for every $x\in\mathbf{R}$,

$$e^x+e^{-x}=2\cdot\left(1+\frac{1}{2!}\cdot x^2+\frac{1}{4!}\cdot x^4+\frac{1}{6!}\cdot x^6+\Lambda+\frac{1}{(2\cdot n)!}\cdot x^{2\cdot n}+\Lambda\right);$$

so, according to the equality (2.2), the equality (7.6) holds.

7) According to the equality (2.5), for every $x\in\mathbf{R}$,

(1) $\mathrm{sch}x\cdot\mathrm{ch}x=1$.

Let be:

(2) $\mathrm{sch}x=a_0+a_1\cdot x+a_2\cdot x^2+a_3\cdot x^3+\Lambda+a_n\cdot x^n+\Lambda$

the expansion in polynomial form of function sch. Then from the equalities (1), (2) and (7.6) it follows that, for every $x\in\mathbf{R}$,

(3) $(a_0+a_1\cdot x+a_2\cdot x^2+a_3\cdot x^3+\Lambda+a_n\cdot x^n+\Lambda)\cdot\left(1+\frac{1}{2!}\cdot x^2+\frac{1}{4!}\cdot x^4+\Lambda+\frac{1}{(2\cdot n)!}\cdot x^{2\cdot n}+\Lambda\right)=1;$

i.e.:

(4)
$$\begin{aligned}
1=\;&a_0+\\
&a_1\cdot x+\\
&\left(\frac{a_0}{2!}+a_2\right)\cdot x^2+\\
&\left(\frac{a_1}{2!}+a_3\right)\cdot x^3+\\
&\left(\frac{a_0}{4!}+\frac{a_2}{2!}+a_4\right)\cdot x^4+\\
&\left(\frac{a_1}{4!}+\frac{a_3}{2!}+a_5\right)\cdot x^5+\\
&\left(\frac{a_0}{6!}+\frac{a_2}{4!}+\frac{a_4}{2!}+a_6\right)\cdot x^6+\\
&\left(\frac{a_1}{6!}+\frac{a_3}{4!}+\frac{a_5}{2!}+a_7\right)\cdot x^7+\\
&\left(\frac{a_0}{8!}+\frac{a_2}{6!}+\frac{a_4}{4!}+\frac{a_6}{2!}+a_8\right)\cdot x^8+\\
&\Lambda\\
&\left(\frac{a_0}{(2\cdot n)!}+\frac{a_2}{(2\cdot n-2)!}+\frac{a_4}{(2\cdot n-4)!}+\Lambda\,\frac{a_{2\cdot n-4}}{4!}+\frac{a_{2\cdot n-2}}{2!}+a_{2\cdot n}\right)\cdot x^{2\cdot n}+\\
&\left(\frac{a_1}{(2\cdot n)!}+\frac{a_3}{(2\cdot n-2)!}+\frac{a_5}{(2\cdot n-4)!}+\Lambda\,\frac{a_{2\cdot n-3}}{4!}+\frac{a_{2\bullet n-1}}{2!}+a_{2\cdot n+1}\right)\cdot x^{2\cdot n+1}+\\
&\Lambda
\end{aligned}$$

Because:

(5) $1=1+0\cdot x+0\cdot x^2+0\cdot x^3+\Lambda+0\cdot x^n+\Lambda$

from the equalities (4) and (5), obtain the following equalities:

(6)
$$\begin{aligned}
&a_0=1\\
&a_1=0\\
&\frac{a_0}{2!}+a_2=0\\
&\frac{a_1}{2!}+a_3=0\\
&\frac{a_0}{4!}+\frac{a_2}{2!}+a_4=0
\end{aligned}$$

$$\frac{a_1}{4!}+\frac{a_3}{2!}+a_5=0$$

$$\frac{a_0}{6!}+\frac{a_2}{4!}+\frac{a_4}{2!}+a_6=0$$

$$\frac{a_1}{6!}+\frac{a_3}{4!}+\frac{a_5}{2!}+a_7=0$$

$$\frac{a_0}{8!}+\frac{a_2}{6!}+\frac{a_4}{4!}+\frac{a_6}{2!}+a_8=0$$

Λ

$$\frac{a_0}{(2\cdot n)!}+\frac{a_2}{(2\cdot n-2)!}+\frac{a_4}{(2\cdot n-4)!}+\Lambda\ \frac{a_{2\cdot n-4}}{4!}+\frac{a_{2\cdot n-2}}{2!}+a_{2\cdot n}=0$$

$$\frac{a_1}{(2\cdot n)!}+\frac{a_3}{(2\cdot n-2)!}+\frac{a_5}{(2\cdot n-4)!}+\Lambda\ \frac{a_{2\cdot n-3}}{4!}+\frac{a_{2\cdot n-1}}{2!}+a_{2\cdot n+1}=0$$

Λ

From the equalities (6), it follows that:

(7) $a_0=1$

$a_1=0$

$a_2=-\frac{1}{2}$

$a_3=0$

$a_4=\frac{5}{24}$

$a_5=0$

$a_6=-\frac{61}{720}$

$a_7=0$

$a_8=\frac{277}{8064}$

Λ

$$a_{2\cdot n}=-\frac{1}{(2\cdot n)!}\cdot[a_0+A_{2\cdot n}^{2}\cdot a_2+A_{2\cdot n}^{4}\cdot a_4+A_{2\cdot n}^{6}\cdot a_6+\Lambda\ +A_{2\cdot n}^{2n-6}\cdot a_{2\cdot n-6}+A_{2\cdot n}^{2\cdot n-4}\cdot a_{2\cdot n-4}+A_{2\cdot n}^{2\cdot n-2}\cdot a_{2\cdot n-2}]$$

$$=\frac{E_{2\cdot n}}{(2\cdot n)!}$$

$a_{2\cdot n+1}=0$

Λ ;

so, according to the equality (2), the equality (7.7) holds. Using a computer can calculate the next values of E_k, for different values of $k\in\mathbf{N}$, such as:

$E_0=1$	$E_2=-1$	$E_4=5$
$E_6=-61$	$E_8=1385$	$E_{10}=-50521$
$E_{12}=2702765$	$E_{14}=-199360981$	$E_{16}=19391512145$
$E_{18}=-2404879675441.$		

8) According to the equality (2.6), for every $x\in\mathbf{R}^*$,

(1) $\mathrm{cshx}\cdot\mathrm{shx}=1.$

Let be:

(2) $\mathrm{cshx}=a_0+a_1\cdot x+a_2\cdot x^2+a_3\cdot x^3+\Lambda\ +a_n\cdot x^n+\Lambda$

the expansion in polynomial form of function csh. Then from the equalities (1), (2) and (7.5) it follows that, for every $x\in\mathbf{R}^*$,

(3) $$(a_0+a_1\cdot x+a_2\cdot x^2+a_3\cdot x^3+\Lambda\ +a_n\cdot x^n+\Lambda\ )\cdot\left(x+\frac{1}{3!}\cdot x^3+\frac{1}{5!}\cdot x^5+\Lambda\ +\frac{1}{(2\cdot n+1)!}\cdot x^{2\cdot n+1}+\Lambda\right)=1.$$

Because:

(4) $1=1+0\cdot x+0\cdot x^2+0\cdot x^3+\Lambda\ +0\cdot x^n+\Lambda$

and in the equality (3) the expression of the left member does not have free term, while that of the right member has free term, from the equalities (4) and (5), obtain that:

(5) $a_0=\frac{1}{x}$

and the equality (3) becomes:

(6) $$\left(\frac{1}{x}+a_1\cdot x+a_2\cdot x^2+\Lambda+a_n\cdot x^n+\Lambda\right)\cdot\left(x+\frac{1}{3!}\cdot x^3+\frac{1}{5!}\cdot x^5+\Lambda+\frac{1}{(2\cdot n+1)!}\cdot x^{2\cdot n+1}+\Lambda\right)=1;$$

i.e.:

(7) $1=1+$

$$\left(\frac{1}{3!}+a_1\right)\cdot x^2+$$

$$a^2\cdot x^3+$$

$$\left(\frac{1}{5!}+\frac{a_1}{3!}+a_3\right)\cdot x^4+$$

$$\left(\frac{a_2}{3!}+a_4\right)\cdot x^5+$$

$$\left(\frac{1}{7!}+\frac{a_1}{5!}+\frac{a_3}{3!}+a_5\right)\cdot x^6+$$

$$\left(\frac{a_2}{5!}+\frac{a_4}{3!}+a_6\right)\cdot x^7+$$

$$\left(\frac{1}{9!}+\frac{a_1}{7!}+\frac{a_3}{5!}+\frac{a_5}{3!}+a_7\right)\cdot x^8+$$

Λ

$$\left(\frac{1}{(2\cdot n+1)!}+\frac{a_1}{(2\cdot n-1)!}+\frac{a_3}{(2\cdot n-3)!}+\frac{a_5}{(2\cdot n-5)!}+\Lambda+\frac{a_{2\cdot n-5}}{5!}+\frac{a_{2\cdot n-3}}{3!}+a_{2n-1}\right)\cdot x^{2\cdot n}+$$

$$\left(\frac{a_2}{(2\cdot n-1)!}+\frac{a_4}{(2\cdot n-3)!}+\frac{a_6}{(2\cdot n-5)!}+\Lambda+\frac{a_{2\cdot n-4}}{5!}+\frac{a_{2\cdot n-2}}{3!}+a_{2\cdot n}\right)\cdot x^{2\cdot n+1}+$$

Λ

From the equalities (7) it follows that:

(8) $\frac{1}{3!}+a_1=0$

$a_2=0$

$\frac{1}{5!}+\frac{a_1}{3!}+a_3=0$

$\frac{a_2}{3!}+a_4=0$

$\frac{1}{7!}+\frac{a_1}{5!}+\frac{a_3}{3!}+a_5=0$

$\frac{a_2}{5!}+\frac{a_4}{3!}+a_6=0$

$\frac{1}{9!}+\frac{a_1}{7!}+\frac{a_3}{5!}+\frac{a_1}{5!}+a_7=0$

Λ

$$\frac{1}{(2\cdot n+1)!}+\frac{a_1}{(2\cdot n-1)!}+\frac{a_3}{(2\cdot n-3)!}+\frac{a_5}{(2\cdot n-5)!}+\Lambda+\frac{a_{2\cdot n-5}}{5!}+\frac{a_{2\cdot n-3}}{3!}+a_{2\cdot n-1}=0$$

$$\frac{a_2}{(2\cdot n-1)!}+\frac{a_4}{(2\cdot n-3)!}+\frac{a_6}{(2\cdot n-5)!}+\Lambda+\frac{a_{2\cdot n-4}}{5!}+\frac{a_{2\cdot n-2}}{3!}+a_{2\cdot n}=0$$

Λ

From the equalities (8), it follows that:

(9) $a_1=-\frac{1}{6}$

$a_2=0$

$a_3=\frac{7}{360}$

$a_4=0$

$a_5=-\frac{31}{15120}$

$a_6=0$

$a_7=\frac{127}{604800}$

Λ

$a_{2\cdot n-1}=-\frac{1}{(2\cdot n+1)!}\cdot[1+A_{2\cdot n+1}^{2}\cdot a_1+A_{2\cdot n+1}^{4}\cdot a_3+A_{2\cdot n+1}^{6}\cdot a_5+\Lambda\ +A_{2\cdot n+1}^{2\cdot n-4}\cdot a_{2\cdot n-5}+A_{2\cdot n+1}^{2\cdot n-2}\cdot a_{2\cdot n-3}]$

$=(-1)^n\cdot\frac{2\cdot(2^{2\cdot n-1}-1)\cdot B_{2\cdot n}}{(2\cdot n)!}$

$a_{2\cdot n}=0$

Λ ;

so, according to the equality (2), the equality (7.8) holds. Using a computer can calculate the next values of B_k, for different values of $k\in\mathbf{N}$, such as:

$B_0=1$	$B_1=\pm\frac{1}{2}$	$B_2=\frac{1}{6}$	$B_3=0$
$B_4=-\frac{1}{30}$	$B_5=0$	$B_6=\frac{1}{42}$	$B_7=0$
$B_8=-\frac{1}{30}$	$B_9=0$	$B_{10}=\frac{5}{66}$	$B_{11}=0$
$B_{12}=-\frac{691}{2730}$	$B_{13}=0$	$B_{14}=\frac{6}{7}$	$B_{15}=0$
$B_{16}=-\frac{3617}{510}$	$B_{17}=0$	$B_{18}=\frac{43867}{798}$	$B_{19}=0$
$B_{20}=-\frac{174611}{330}$.			

9) According to the equalities (2.3) and (2.5), for every $x\in\mathbf{R}$,

(1) thx=shx·schx.

From the equalities (1), (7.5) and (7.7), it follows that, for every $x\in(-1,1)$:

(2) $\text{thx}=\left(\frac{1}{1!}\cdot x+\frac{1}{3!}\cdot x^3+\frac{1}{5!}\cdot x^5+\Lambda\ +\frac{1}{(2\cdot n+1)!}\cdot x^{2\cdot n+1}+\Lambda\right)\cdot$

$\left(1-\frac{1}{2}\cdot x^2+\frac{5}{24}\cdot x^4-\frac{61}{720}\cdot x^6+\frac{277}{8064}\cdot x^8-\Lambda\ +(-1)^n\cdot\frac{E_{2\cdot n}}{2\cdot n}\cdot x^{2\cdot n}+\Lambda\right)$

$=\left(\sum_{n=0}^{\infty}\frac{1}{(2\cdot n+1)!}\cdot x^{2\cdot n+1}\right)\cdot\left(\sum_{n=0}^{\infty}(-1)^n\cdot\frac{E_{2\cdot n}}{(2\cdot n)!}\cdot x^{2\cdot n}\right)$

$=x+$

$\left(\frac{1}{3!}-\frac{1}{1!}\cdot\frac{1}{2}\right)\cdot x^3+$

$\left(\frac{1}{1!}\cdot\frac{5}{24}-\frac{1}{3!}\cdot\frac{1}{2}+\frac{1}{5!}\cdot\frac{1}{1}\right)\cdot x^5+$

$\left(-\frac{1}{1!}\cdot\frac{61}{720}+\frac{1}{3!}\cdot\frac{5}{24}-\frac{1}{5!}\cdot\frac{1}{2}+\frac{1}{7!}\cdot\frac{1}{1}\right)\cdot x^7+$

$\left(\frac{1}{1!}\cdot\frac{277}{8064}-\frac{1}{3!}\cdot\frac{61}{720}+\frac{1}{5!}\cdot\frac{5}{24}-\frac{1}{7!}\cdot\frac{1}{2}+\frac{1}{9!}\cdot\frac{1}{1}\right)\cdot x^9+$

Λ

$a_{2\cdot n-1}\cdot x^{2\cdot n-1}+$

Λ ,

where, for every $n\in\mathbf{N}^*$,

$$a_{2\cdot n-1}=\frac{1}{1!}\cdot(-1)^{n-1}\cdot\frac{E_{2\cdot n-2}}{(2\cdot n-2)!}+\frac{1}{3!}\cdot(-1)^{n-2}\cdot\frac{E_{2\cdot n-4}}{(2\cdot n-4)!}+\frac{1}{5!}\cdot(-1)^{n-3}\cdot\frac{E_{2\cdot n-6}}{(2\cdot n-6)!}+\Lambda+$$
$$\frac{1}{(2\cdot n-5)!}\cdot(-1)^{2}\cdot\frac{E_4}{4!}+\frac{1}{(2\cdot n-3)!}\cdot(-1)^{1}\cdot\frac{E_2}{2!}+\frac{1}{(2\cdot n-1)!}\cdot(-1)^{0}\cdot\frac{E_0}{0!}$$
$$=\frac{2^{2\cdot n}\cdot(2^{2\cdot n}-1)\cdot B_{2\cdot n}}{(2\cdot n)!},$$

Because, it proves easy: for every $n\in\mathbf{N}^*$,

(3) $$B_{2\cdot n}=\sum_{k=0}^{n-1}C_{2\cdot n-1}^{2\cdot k}\cdot\frac{2\cdot n}{4^{2\cdot n}-2^{2\cdot n}}\cdot E_{2\cdot k},$$

or, in other words,

(4) $$E_{2\cdot n}=\sum_{k=1}^{2\cdot n}C_{2\cdot n}^{k-1}\cdot\frac{2^k-4^k}{k}\cdot B_k,$$

By performing calculations, from the equalities (2), obtain that:

(5) $$\text{thx}=x-\frac{1}{3}\cdot x^3+\frac{2}{15}\cdot x^5-\frac{17}{315}\cdot x^7+\frac{62}{2835}\cdot x^9+\Lambda+\frac{2^{2\cdot n}\cdot(2^{2\cdot n}-1)\cdot B_{2\cdot n}}{(2\cdot n)!}\cdot x^{2\cdot n-1}+\Lambda\ ,$$

i.e. the equalities (7.9) hold.

10) According to the equalities (2.4) and (2.6), for every $x\in\mathbf{R}^*$,

(1) cthx=chx·cshx.

From the equalities (1), (7.6) and (7.8), it follows that:

(2) $$\text{cthx}=\left(1+\frac{1}{2!}\cdot x^2+\frac{1}{4!}\cdot x^4+\frac{1}{6!}\cdot x^6+\frac{1}{8!}\cdot x^8+\Lambda+\frac{1}{(2\cdot n)!}\cdot x^{2\cdot n}+\Lambda\right)\cdot$$
$$\left(\frac{1}{x}-\frac{1}{6}\cdot x+\frac{7}{360}\cdot x^3-\frac{31}{15120}\cdot x^5+\frac{127}{604800}\cdot x^7+\Lambda+(-1)^n\cdot\frac{2\cdot(2^{2\cdot n-1}-1)\cdot B_{2\cdot n}}{(2\cdot n)!}\cdot x^{2\cdot n-1}+\Lambda\right)$$
$$=\left(\sum_{n=0}^{\infty}\frac{1}{(2\cdot n)!}\cdot x^{2\cdot n}\right)\cdot\left(\frac{1}{x}+\sum_{n=1}^{\infty}(-1)^n\cdot\frac{2\cdot(2^{2\cdot n-1}-1)\cdot B_{2\cdot n}}{(2\cdot n)!}\cdot x^{2\cdot n-1}\right)$$
$$=\frac{1}{x}+$$
$$\left(\frac{1}{2!}-\frac{1}{6}\right)\cdot x+$$
$$\left(\frac{7}{360}-\frac{1}{2!}\cdot\frac{1}{6}+\frac{1}{4!}\right)\cdot x^3+$$
$$\left(-\frac{31}{15120}+\frac{1}{2!}\cdot\frac{7}{360}-\frac{1}{4!}\cdot\frac{1}{6}+\frac{1}{6!}\right)\cdot x^5+$$
$$\left(\frac{127}{604800}-\frac{1}{2!}\cdot\frac{31}{15120}+\frac{1}{4!}\cdot\frac{7}{360}-\frac{1}{6!}\cdot\frac{1}{6}+\frac{1}{8!}\right)\cdot x^7+$$
$$\Lambda$$
$$a_{2\cdot n-1}\cdot x^{2\cdot n-1}+$$
$$\Lambda\ ,$$

where, for every $n\in\mathbf{N}^*$,

$$a_{2\cdot n-1}=\frac{(-1)^n}{0!}\cdot\frac{2\cdot(2^{2\cdot n-1}-1)\cdot B_{2\cdot n}}{(2\cdot n)!}+\frac{(-1)^{n-1}}{2!}\cdot\frac{2\cdot(2^{2\cdot n-3}-1)\cdot B_{2\cdot n-2}}{(2\cdot n-2)!}+$$
$$\frac{(-1)^{n-2}}{4!}\cdot\frac{2\cdot(2^{2\cdot n-5}-1)\cdot B_{2\cdot n-4}}{(2\cdot n-4)!}+\Lambda+\frac{(-1)^3}{(2\cdot n-6)!}\cdot\frac{2\cdot(2^5-1)\cdot B_6}{6!}+$$
$$\frac{(-1)^2}{(2\cdot n-4)!}\cdot\frac{2\cdot(2^3-1)\cdot B_4}{4!}+\frac{(-1)^1}{(2\cdot n-2)!}\cdot\frac{2\cdot(2^2-1)\cdot B_2}{2!}$$

$$=\frac{2^{2\cdot n}\cdot B_{2\cdot n}}{(2\cdot n)!},$$

Because, from the equality (3) from the previous point, it follows that: for every $n\in\mathbf{N}^*$,

(3) $B_n=1-\sum_{k=0}^{n-1} C_{n-1}^{k}\cdot\frac{B_k}{n-k+1}$.

By performing calculations, from the equalities (2), obtain that:

(4) $cthx=\frac{1}{x}+\frac{1}{3}\cdot x-\frac{1}{45}\cdot x^3+\frac{2}{945}\cdot x^5-\frac{1}{4725}\cdot x^7+\Lambda+\frac{2^{2\cdot n}\cdot B_{2\cdot n}}{(2\cdot n)!}\cdot x^{2\cdot n-1}+\Lambda$,

which shows that the equalities (7.10) hold.

***Otherwise*:** According to the equalities (2.3) and (2.4), for every $x\in\mathbf{R}^*$,

(5) $thx\cdot cthx=1$.

Let be:

(6) $cthx=a_0+a_1\cdot x+a_2\cdot x^2+a_3\cdot x^3+\Lambda+a_n\cdot x^n+\Lambda$

the expansion in polynomial form of function cth. Then from the equalities (1), (2) and (7.9) it follows that, for every $x\in\mathbf{R}^*$,

(7) $(a_0+a_1\cdot x+a_2\cdot x^2+a_3\cdot x^3+\Lambda+a_n\cdot x^n+\Lambda)\cdot$

$$\left(x-\frac{1}{3}\cdot x^3+\frac{2}{15}\cdot x^5-\frac{17}{315}\cdot x^7+\frac{62}{2835}\cdot x^9-\Lambda+\frac{2^{2\cdot n}\cdot(2^{2\cdot n}-1)\cdot B_{2\cdot n}}{(2\cdot n)!}\cdot x^{2\cdot n-1}+\Lambda\right)=1;$$

i.e.:

(8) $1=a_0\cdot x+$

$a_1\cdot x^2+$

$\left(-\frac{a_0}{3}+a_2\right)\cdot x^3+$

$\frac{a_1}{3}\cdot x^4+$

$\left(\frac{2\cdot a_0}{15}-\frac{a_2}{3}+a_4\right)\cdot x^5+$

$\left(\frac{2\cdot a_1}{15}-\frac{a_3}{3}+a_5\right)\cdot x^6+$

$\left(-\frac{17\cdot a_0}{315}+\frac{2\cdot a_2}{15}-\frac{a_4}{3}+a_6\right)\cdot x^7+$

$\left(-\frac{17\cdot a_1}{315}+\frac{2\cdot a_3}{15}-\frac{a_5}{3}+a_7\right)\cdot x^8+$

$\left(\frac{62\cdot a_0}{2835}-\frac{17\cdot a_2}{315}+\frac{2\cdot a_4}{15}-\frac{a_6}{3}+a_8\right)\cdot x^9+$

Λ

$(a_0\cdot b_{2\cdot n-1}+a_1\cdot b_{2\cdot n-2}+\Lambda+a_{2\cdot n-2}\cdot b_1+a_{2\cdot n-1}\cdot b_0)\cdot x^{2\cdot n}+$

$(a_0\cdot b_{2\cdot n}+a_1\cdot b_{2\cdot n-1}+\Lambda+a_{2\cdot n-1}\cdot b_1+a_{2\cdot n}\cdot b_0)\cdot x^{2\cdot n+1}+$

Λ,

where, for every $n\in\mathbf{N}$,

$$b_{2\cdot n}=\frac{2^{2\cdot n}\cdot B_{2\cdot n}}{(2\cdot n)!}$$

and

$$b_{2\cdot n+1}=0.$$

Further think as in the proof of equalities (7.6) and obtain the same results as above.

11) Let $\alpha\in\mathbf{R}^*$ be a not null real number, fixed and considers the function:

$f:(-1,1)\to\mathbf{R}$,

defined by: for every $x\in(-1,1)$,

$f(x)=(1+x)^{\alpha}$.

Then, for every $x\in(-1,1)$,

$$f'(x)=\alpha\cdot(1+x)^{\alpha-1},$$
$$f''(x)=\alpha\cdot(\alpha-1)\cdot(1+x)^{\alpha-2},$$
$$f'''(x)=\alpha\cdot(\alpha-1)\cdot(\alpha-2)\cdot(1+x)^{\alpha-3},$$
$$\Lambda$$
$$f^{(n)}(x)=\alpha\cdot(\alpha-1)\cdot(\alpha-2)\cdot\Lambda\cdot(\alpha-n+2)\cdot(\alpha-n+1)\cdot(1+x)^{\alpha-n},$$
$$\Lambda\ ,$$

and, thus:

$$f'(0)=\alpha,$$
$$f''(0)=\alpha\cdot(\alpha-1),$$
$$f'''(0)=\alpha\cdot(\alpha-1)\cdot(\alpha-2),$$
$$\Lambda$$
$$f^{(n)}(0)=\alpha\cdot(\alpha-1)\cdot(\alpha-2)\cdot\Lambda\cdot(\alpha-n+2)\cdot(\alpha-n+1)\cdot,$$
$$\Lambda\ ,$$

and:

(1) $$(1+x)^{\alpha}=1+\frac{1}{1!}\cdot\alpha\cdot x+\frac{1}{2!}\cdot\alpha\cdot(\alpha-1)\cdot x^2+\frac{1}{3!}\cdot\alpha\cdot(\alpha-1)\cdot(\alpha-2)\cdot x^3+\Lambda+\frac{1}{n!}\cdot\alpha\cdot(\alpha-1)\cdot(\alpha-2)\cdot\Lambda\cdot(\alpha-n+2)\cdot(\alpha-n+1)\cdot x^n+\Lambda$$

From the equalities (1) it follows that, for every $x\in(-1,1)$,

(2) $$(1-x)^{\alpha}=1-\frac{1}{1!}\cdot\alpha\cdot x+\frac{1}{2!}\cdot\alpha\cdot(\alpha-1)\cdot x^2-\frac{1}{3!}\cdot\alpha\cdot(\alpha-1)\cdot(\alpha-2)\cdot x^3+\Lambda+(-1)^n\cdot\frac{1}{n!}\cdot\alpha\cdot(\alpha-1)\cdot(\alpha-2)\cdot\Lambda\cdot(\alpha-n+2)\cdot(\alpha-n+1)\cdot x^n+\Lambda$$

and:

(3) $$(1+x^2)^{\alpha}=1+\frac{1}{1!}\cdot\alpha\cdot x^2+\frac{1}{2!}\cdot\alpha\cdot(\alpha-1)\cdot x^4+\frac{1}{3!}\cdot\alpha\cdot(\alpha-1)\cdot(\alpha-2)\cdot x^6+\Lambda+\frac{1}{n!}\cdot\alpha\cdot(\alpha-1)\cdot(\alpha-2)\cdot\Lambda\cdot(\alpha-n+2)\cdot(\alpha-n+1)\cdot x^{2n}+\Lambda\ ,$$

respective,

(4) $$(1-x^2)^{\alpha}=1-\frac{1}{1!}\cdot\alpha\cdot x^2+\frac{1}{2!}\cdot\alpha\cdot(\alpha-1)\cdot x^4-\frac{1}{3!}\cdot\alpha\cdot(\alpha-1)\cdot(\alpha-2)\cdot x^6+\Lambda+(-1)^n\cdot\frac{1}{n!}\cdot\alpha\cdot(\alpha-1)\cdot(\alpha-2)\cdot\Lambda\cdot(\alpha-n+2)\cdot(\alpha-n+1)\cdot x^{2\cdot n}+\Lambda\ .$$

On the other hand, from the equalities (3) and (4.49), it follows that, for every $x\in(-1,1)$,

(5) $$(sh^{-1}x)'=\frac{1}{\sqrt{x^2+1}}=(1+x^2)^{-\frac{1}{2}}$$
$$=1+\frac{1}{1!}\cdot\left(-\frac{1}{2}\right)\cdot x^2+\frac{1}{2!}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot x^4+$$
$$\frac{1}{3!}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot x^6+\Lambda+$$
$$\frac{1}{n!}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\Lambda\cdot\left(-\frac{2\cdot n-3}{2}\right)\cdot\left(-\frac{2\cdot n-1}{2}\right)\cdot x^{2\cdot n}+\Lambda+$$
$$=1-\left(\frac{1}{2}\right)\cdot x^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot x^4-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot x^6+\Lambda+\left(\frac{1\cdot 3\cdot 5\cdot\Lambda\cdot(2\cdot n-3)\cdot(2\cdot n-1)}{2\cdot 4\cdot 6\cdot\Lambda\cdot(2\cdot n-2)\cdot(2\cdot n)}\right)\cdot x^{2\cdot n}+\Lambda\ .$$

From the equalities (5), by integration, it follows that, for every $x\in(-1,1)$,

$$sh^{-1}x=\int_0^x\frac{1}{\sqrt{t^2+1}}\cdot dt$$
$$=\int_0^x\left[1-\left(\frac{1}{2}\right)\cdot x^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot x^4-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 7}\right)\cdot x^6+\Lambda+\left(\frac{1\cdot 3\cdot\Lambda\cdot(2\cdot n-1)}{2\cdot 4\cdot\Lambda\,(2\cdot n)}\right)\cdot x^{2\cdot n}+\Lambda\right]\cdot dt$$

$$=x-\left(\frac{1}{2}\right)\cdot\frac{x^3}{3}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{x^5}{5}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{x^7}{7}+\Lambda+\left(\frac{1\cdot 3\cdot 5\cdot\Lambda\cdot(2\cdot n-1)}{2\cdot 4\cdot 6\cdot\Lambda\cdot(2\cdot n)}\right)\cdot\frac{x^{2\cdot n+1}}{2\cdot n+1}+\Lambda\ .$$

$$=\sum_{n=0}^{\infty}\left(\frac{(-1)^n\cdot(2\cdot n)!}{2^{2n}\cdot(n!)^2}\right)\cdot\frac{x^{2\cdot n+1}}{2\cdot n+1}\ ;$$

i.e., the equalities (7.11) hold. Now, from the equalities (3) and (4.49), it follows that, for every $x\in(1,+\infty)$,

(6) $$(sh^{-1}x)'=\frac{1}{\sqrt{x^2+1}}=\frac{\frac{1}{x}}{\frac{\sqrt{x^2+1}}{x}}=\frac{\frac{1}{x}}{\sqrt{\frac{1}{x^2}+1}}=\frac{1}{x}\cdot\left(1+\frac{1}{x^2}\right)^{-\frac{1}{2}}$$

$$=\frac{1}{x}\cdot[1+\frac{1}{1!}\cdot\left(-\frac{1}{2}\right)\cdot\frac{1}{x^2}+\frac{1}{2!}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\frac{1}{x^4}+$$

$$\frac{1}{3!}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\frac{1}{x^6}+\Lambda+$$

$$\frac{1}{n!}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\Lambda\cdot\left(-\frac{2\cdot n-3}{2}\right)\cdot\left(-\frac{2\cdot n-1}{2}\right)\cdot\frac{1}{x^{2\cdot n}}+\Lambda\ ]$$

$$=\frac{1}{x}-\left(\frac{1}{2}\right)\cdot\frac{1}{x^3}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{x^5}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{1}{x^7}+\Lambda+$$

$$\left(\frac{1\cdot 3\cdot 5\cdot\Lambda\cdot(2\cdot n-3)\cdot(2\cdot n-1)}{2\cdot 4\cdot 6\cdot\Lambda\cdot(2\cdot n-2)\cdot(2\cdot n)}\right)\cdot\frac{1}{x^{2\cdot n+1}}+\Lambda\ .$$

From the equalities (6), by integration, it follows that, for every $x\in(-1,1)$,

$$sh^{-1}x=\int_0^x\frac{1}{\sqrt{t^2+1}}\cdot dt$$

$$=\int_0^x\left[\frac{1}{x}-\left(\frac{1}{2}\right)\cdot\frac{1}{x^3}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{x^5}-\Lambda+\left(\frac{1\cdot 3\cdot\Lambda\cdot(2\cdot n-1)}{2\cdot 4\cdot\Lambda\ (2\cdot n)}\right)\cdot\frac{1}{x^{2\cdot n+1}}+\Lambda\right]\cdot dt$$

$$=\ln(2\cdot x)+\left(\frac{1}{2}\right)\cdot\frac{1}{2\cdot x^2}-\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{4\cdot x^4}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{1}{6\cdot x^6}-\Lambda+$$

$$(-1)^{n-1}\cdot\left(\frac{1\cdot 3\cdot 5\cdot\Lambda\cdot(2\cdot n-1)}{2\cdot 4\cdot 6\cdot\Lambda\cdot(2\cdot n)}\right)\cdot\frac{1}{(2\cdot n)\cdot x^{2\cdot n}}+\Lambda$$

$$=\ln(2\cdot x)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot(2\cdot n-1)!!}{2\cdot n\cdot(2\cdot n)!!}\cdot\frac{1}{x^{2\cdot n}}\ ;$$

which shows that the equalities (7.12) hold. Finally, for $x\in(-\infty,-1)$, also from the equalities (3) and (4.49), it follows that,

(7) $$(sh^{-1}x)'=\frac{1}{\sqrt{x^2+1}}=-\frac{\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}}=-\frac{1}{x}\cdot\left(1+\frac{1}{x^2}\right)^{-\frac{1}{2}}\ ;$$

and, further, to obtain the equalities (7.13) we proceed as with equalities (7.12).

12) From the equalities (4) from the proof of statements from point 11) and (4.50), it follows that, for every $x\in(1,+\infty)$,

(1) $$(ch_1^{-1}x)'=-\frac{1}{\sqrt{x^2-1}}=-\frac{\frac{1}{x}}{\sqrt{1-\frac{1}{x^2}}}=-\frac{1}{x}\cdot\left(1-\frac{1}{x^2}\right)^{-\frac{1}{2}}$$

$$=-\frac{1}{x}\cdot[1-$$

$$\frac{1}{1!}\cdot\left(-\frac{1}{2}\right)\cdot\frac{1}{x^2}+$$
$$\frac{1}{2!}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\frac{1}{x^4}-$$
$$\frac{1}{3!}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\frac{1}{x^6}-$$
$$\Lambda +$$
$$\frac{1}{n!}\cdot(-1)^n\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\Lambda\cdot\left(-\frac{2\cdot n-3}{2}\right)\cdot\left(-\frac{2\cdot n-1}{2}\right)\cdot\frac{1}{x^{2\cdot n}}+$$
$$\Lambda\ ]$$
$$=-\frac{1}{x}-\left(\frac{1}{2}\right)\cdot\frac{1}{x^3}-\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{x^5}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{1}{x^7}-\Lambda -$$
$$\left(\frac{1\cdot 3\cdot 5\cdot\Lambda\cdot(2\cdot n-3)\cdot(2\cdot n-1)}{2\cdot 4\cdot 6\cdot\Lambda\cdot(2\cdot n-2)\cdot(2\cdot n)}\right)\cdot\frac{1}{x^{2\cdot n+1}}+\Lambda\ .$$

From the equalities (1), by integration, it follows that, for every $x\in(1,+\infty)$,

(2) $$ch_1^{-1}x=-\int_0^x\frac{1}{\sqrt{t^2-1}}\cdot dt$$
$$=\int_0^x\left[-\frac{1}{x}-\left(\frac{1}{2}\right)\cdot\frac{1}{x^3}-\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{x^5}-\Lambda-\left(\frac{1\cdot 3\cdot\Lambda\cdot(2\cdot n-1)}{2\cdot 4\cdot\Lambda\ (2\cdot n)}\right)\cdot\frac{1}{x^{2\cdot n+1}}+\Lambda\right]\cdot dt$$
$$=-\ln(2x)-\left(\frac{1}{2}\right)\cdot\frac{1}{2\cdot x^2}-\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{4\cdot x^4}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{1}{6\cdot x^6}-\Lambda -$$
$$\left(\frac{1\cdot 3\cdot 5\cdot\Lambda\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\Lambda\cdot(2n)}\right)\cdot\frac{1}{(2n)\cdot x^{2n}}+\Lambda$$
$$=-\ln(2\cdot x)-\sum_{k=1}^{\infty}\frac{(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\cdot\frac{1}{(2\cdot n)\cdot x^{2\cdot n}};$$

i.e., the equalities (7.14) hold. On the other hand, from the equalities (4) from the proof of statements from point 11) and (4.50′), it follows that, for every $x\in(1,+\infty)$,

(3) $$(ch_2^{-1}x)'=\frac{1}{\sqrt{x^2-1}}=\frac{\frac{1}{x}}{\frac{\sqrt{x^2-1}}{x}}=\frac{\frac{1}{x}}{\sqrt{1-\frac{1}{x^2}}}=\frac{1}{x}\cdot\left(1-\frac{1}{x^2}\right)^{-\frac{1}{2}}$$
$$=\frac{1}{x}\cdot\left[1-\frac{1}{1!}\cdot\left(-\frac{1}{2}\right)\cdot\frac{1}{x^2}+\frac{1}{2!}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\frac{1}{x^4}-\right.$$
$$\frac{1}{3!}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\frac{1}{x^6}+\Lambda +$$
$$\left.\frac{1}{n!}\cdot(-1)^n\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\Lambda\cdot\left(-\frac{2\cdot n-3}{2}\right)\cdot\left(-\frac{2\cdot n-1}{2}\right)\cdot\frac{1}{x^{2\cdot n}}+\Lambda\right]$$
$$=\frac{1}{x}+\left(\frac{1}{2}\right)\cdot\frac{1}{x^3}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{x^5}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{1}{x^7}+\Lambda +$$
$$\left(\frac{1\cdot 3\cdot 5\cdot\Lambda\cdot(2\cdot n-3)\cdot(2\cdot n-1)}{2\cdot 4\cdot 6\cdot\Lambda\cdot(2\cdot n-2)\cdot(2\cdot n)}\right)\cdot\frac{1}{x^{2\cdot n+1}}+\Lambda\ .$$

From the equalities (3), by integration, it follows that, for every $x\in(1,+\infty)$,

$$ch_2^{-1}x=\int_0^x\frac{1}{\sqrt{t^2+1}}\cdot dt$$

$$=\int_0^x\left[\frac{1}{x}+\left(\frac{1}{2}\right)\cdot\frac{1}{x^3}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{x^5}+\Lambda+\left(\frac{1\cdot 3\cdot\Lambda\cdot(2\cdot n-1)}{2\cdot 4\cdot\Lambda\ (2\cdot n)}\right)\cdot\frac{1}{x^{2\cdot n+1}}+\Lambda\right]\cdot dt$$

$$=\ln(2x)+\left(\frac{1}{2}\right)\cdot\frac{1}{2\cdot x^2}-\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{4\cdot x^4}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{1}{6\cdot x^6}+\Lambda+$$

$$\left(\frac{1\cdot 3\cdot 5\cdot\Lambda\cdot(2\cdot n-1)}{2\cdot 4\cdot 6\cdot\Lambda\cdot(2\cdot n)}\right)\cdot\frac{1}{(2\cdot n)\cdot x^{2\cdot n}}+\Lambda$$

$$=\ln(2\cdot x)+\sum_{n=1}^{\infty}\left(\frac{(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\right)\cdot\frac{1}{(2\cdot n)\cdot x^{2\cdot n}};$$

which shows that the equalities (7.15) hold.

13) According to the equality (3.25), for every $x\in(-1,1)$,

(1) $$th^{-1}x=\frac{1}{2}\cdot\ln\left(\frac{1+x}{1-x}\right)$$

$$=\frac{1}{2}\cdot[\ln(1+x)-\ln(1-x)].$$

From the equality (1) from the proof of statements from point 11), above, it follows that, for every $x\in(-1,1)$,

(2) $$\frac{1}{1+x}=(1+x)^{-1}$$

$$=1+\frac{1}{1!}\cdot(-1)\cdot x+\frac{1}{2!}\cdot(-1)\cdot(-1-1)\cdot x^2+\frac{1}{3!}\cdot(-1)\cdot(-1-1)\cdot(-1-2)\cdot x^3+\Lambda+$$

$$\frac{1}{n!}\cdot(-1)\cdot(-1-1)\cdot(-1-2)\cdot\Lambda\cdot(-1-n+2)\cdot(-1-n+1)\cdot x^n+\Lambda,$$

i.e.:

(3) $$\frac{1}{1+x}=(1+x)^{-1}$$

$$=1+\frac{1}{1!}\cdot(-1)\cdot x+\frac{1}{2!}\cdot(-1)^2\cdot 2!\cdot x^2+\frac{1}{3!}\cdot(-1)^3\cdot 3!\cdot x^3+\Lambda+\frac{1}{n!}\cdot(-1)^n\cdot n!\cdot x^n+\Lambda,$$

$$=1-x+x^2-x^3+\Lambda+(-1)^n\cdot x^n+\Lambda.$$

From the equality (3), by integration, obtain that, for every $x\in(-1,1)$:

(4) $$\ln(1+x)=\int_0^x\frac{1}{1+t}\cdot dt$$

$$=\int_0^x(1-x+x^2-x^3+\Lambda+(-1)^n\cdot x^n+\Lambda)\cdot dt$$

$$=x-\frac{1}{2}\cdot x^2+\frac{1}{3}\cdot x^3+\Lambda+\frac{1}{n}\cdot x^n+\Lambda.$$

In an analogous manner obtain that, for every $x\in(-1,1)$,

(5) $$\frac{1}{1-x}=(1-x)^{-1}$$

$$=1-\frac{1}{1!}\cdot(-1)\cdot x+\frac{1}{2!}\cdot(-1)^2\cdot 2!\cdot x^2-\frac{1}{3!}\cdot(-1)^3\cdot 3!\cdot x^3+\Lambda+\frac{1}{n!}\cdot(-1)^{2n}\cdot n!\cdot x^n+\Lambda,$$

$$=1+x+x^2+x^3+\Lambda+\cdot x^n+\Lambda.$$

and from the equality (5), by integration, obtain that, for every $x\in(-1,1)$:

(6) $$\ln(1-x)=-\int_0^x\frac{1}{1-t}\cdot dt=-\int_0^x(1+x+x^2+x^3+\Lambda+x^n+\Lambda)\cdot dt$$

$$=-x-\frac{1}{2}\cdot x^2-\frac{1}{3}\cdot x^3-\Lambda-\frac{1}{n}\cdot x^n-\Lambda.$$

By decreasing member with member of the equalities (4) and (6), it follows that:

(7) $$\ln\left(\frac{1+x}{1-x}\right)=2\cdot\left(x+\frac{1}{3}\cdot x^3+\frac{1}{5}\cdot x^5+\Lambda+\frac{1}{n}\cdot x^n+\Lambda\right),$$

whence, according to the equality (1), obtain the equality (7.16).

14) The equality (7.17) follows from the equalities (5.20) and (7.16), replacing on x with $\frac{1}{x}$.

15) From the equalities (5.18) and (7.14), it follows that, for every $x\in(0,1)$,

$$\operatorname{sch}_1^{-1} x=\operatorname{ch}_1^{-1}\left(\frac{1}{x}\right)$$

$$=-\ln\left(\frac{2}{x}\right)-\left[\left(\frac{1}{2}\right)\cdot\frac{x^2}{2}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{x^4}{4}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{x^6}{6}+\Lambda\right]$$

$$=-\ln\left(\frac{2}{x}\right)-\sum_{n=1}^{\infty}\frac{(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\cdot\frac{x^{2\cdot n}}{(2\cdot n)};$$

so the equalities (7.18) hold. On the other hand, from the equalities (5.19) and (7.15), it follows that, for every $x\in(0,1)$,

$$\operatorname{sch}_2^{-1} x=\operatorname{ch}_2^{-1}\left(\frac{1}{x}\right)$$

$$=\ln\left(\frac{2}{x}\right)+\left[\left(\frac{1}{2}\right)\cdot\frac{x^2}{2}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{x^4}{4}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{x^6}{6}+\Lambda\right]$$

$$=\ln\left(\frac{2}{x}\right)+\sum_{n=1}^{\infty}\frac{(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\cdot\frac{x^{2\cdot n}}{(2\cdot n)};$$

which shows that the equalities from (7.19) hold also.

16) From the equalities (5.16) and (7.11), it follows that, for every $x\in(-\infty,-1)$,

$$\operatorname{csh}_1^{-1} x=\operatorname{sh}^{-1}\left(\frac{1}{x}\right)$$

$$=\frac{1}{x}-\left(\frac{1}{2}\right)\cdot\frac{1}{3\cdot x^3}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{5\cdot x^5}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{1}{7\cdot x^7}+\Lambda+$$

$$\frac{(-1)^n\cdot(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\cdot\frac{1}{(2\cdot n+1)\cdot x^{2\cdot n+1}}+\Lambda$$

$$=\sum_{n=0}^{\infty}\left(\frac{(-1)^n\cdot(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\right)\cdot\frac{1}{(2\cdot n+1)\cdot x^{2\cdot n+1}};$$

which shows that the equalities (7.20) hold. Then, from the equalities (5.16) and (7.13), it follows that, for every $x\in(-1,0)$,

$$\operatorname{csh}_1^{-1} x=\operatorname{sh}^{-1}\left(\frac{1}{x}\right)$$

$$=-\ln\left(\frac{2}{|x|}\right)-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot(2\cdot n-1)!!}{(2\cdot n)!!}\cdot\frac{x^{2\cdot n}}{2\cdot n};$$

so the equalities (7.21) hold also. Further, from the equalities (5.17) and (7.11), it follows that, for every $x\in(0,1)$,

$$\operatorname{csh}_2^{-1} x=\operatorname{sh}^{-1}\left(\frac{1}{x}\right)$$

$$=\ln\left(\frac{2}{x}\right)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot(2\cdot n-1)!!}{(2\cdot n)!!}\cdot\frac{x^{2\cdot n}}{2\cdot n};$$

therefore, (also) the equalities (7.22) hold. Finally, from the equalities (5.17) and (7.12), it follows that, for every $x\in(1,+\infty)$,

$$\operatorname{csh}_2^{-1} x=\operatorname{sh}^{-1}\left(\frac{1}{x}\right)$$

$$=\frac{1}{x}-\left(\frac{1}{2}\right)\cdot\frac{1}{3\cdot x^3}+\left(\frac{1\cdot 3}{2\cdot 4}\right)\cdot\frac{1}{5\cdot x^5}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\cdot\frac{1}{7\cdot x^7}+\Lambda+$$

$$\frac{(-1)^n\cdot(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\cdot\frac{1}{(2\cdot n+1)\cdot x^{2\cdot n+1}}+\Lambda$$

$$=\sum_{n=0}^{\infty}\left(\frac{(-1)^n\cdot(2\cdot n)!}{2^{2\cdot n}\cdot(n!)^2}\right)\cdot\frac{1}{(2\cdot n+1)\cdot x^{2\cdot n+1}};$$

i.e. the equalities (7.23) are true, too. □

III. Conclusions and additions

Here we come to the end with the properties of hyperbolic functions. In total, we presented, in these five works, 176 properties, expressed in 185 equalities. Of course, the question arises: "*I presented all the properties of these functions.*" Categorical answer: No. Sometimes, after publishing a list of such properties, I immediately found others. For example, the group of properties E - Other trigonometric properties, can be supplemented by the following:

42′) *For every x, y∈**R**, the following equalities hold*:

1) $ch^2x+sh^2y=sh^2x+ch^2y=ch(x+y)\cdot ch(x-y)$ (4.42′)

2) $ch^2x-ch^2y=sh^2x-sh^2y=sh(x+y)\cdot sh(x-y)$. (4.42″)

42″) *For every x∈**R** and every n∈**N**, the following equalities hold*:

1) $(chx+shx)^n=chnx+shnx$; (4.42‴)

2) $(chx-shx)^n=chnx-shnx$; (4.42$^{(iv)}$)

3) $ch^nx=\frac{1}{2^n}\cdot[chnx+C_n^1\cdot ch(n-2)x+\Lambda]$; (4.42$^{(v)}$)

4) $sh^nx=\frac{1}{2^n}\cdot[shnx-C_n^1\cdot sh(n-2)x+\Lambda]$; *(n - impar)* (4.42$^{(vi)}$)

5) $chnx=ch^nx+C_n^2\cdot ch^{n-2}x\cdot sh^2x+\Lambda$; (4.42$^{(vii)}$)

6) $shnx=C_n^1\cdot ch^{n-1}x\cdot shx+C_n^3\cdot ch^{n-3}x\cdot sh^3x+\Lambda$. (4.42$^{(viii)}$)

***Hints*: 42′)1)** We use the equalities (3.4), (3.5) and (3.1); or (4.33), (4.4) and (3.1).

2) We use the equalities: (3.2), (3.3) and (3.1); or (4.34), (4.4) and (3.1).

42″)1) We use mathematical induction and equalities: (4.2), (4.4), (3.2) and (3.4).

2) We use mathematical induction and equalities: (4.2), (4.4), (3.3) and (3.5).

3) Let be x∈**R**:

$\alpha=chx+shx$ and $\beta=chx-shx$.

Then, according to the equalities (4.42‴) and (4.42$^{(iv)}$),

$$2^n\cdot ch^nx=\left(\alpha+\frac{1}{\alpha}\right)^n$$
$$=\alpha^n+C_n^1\cdot\alpha^{n-2}+C_n^2\cdot\alpha^{n-4}+\Lambda$$
$$=[chnx+shnx]+C_n^1\cdot[ch(n-2)x+sh(n-2)x]+C_n^2\cdot[ch(n-4)x+sh(n-4)x]+\Lambda$$

and:

$$2^n\cdot ch^nx=\left(\beta+\frac{1}{\beta}\right)^n$$
$$=\beta^n+C_n^1\cdot\beta^{n-2}+C_n^2\cdot\beta^{n-4}+\Lambda$$
$$=[chnx-shnx]+C_n^1\cdot[ch(n-2)x-sh(n-2)x]+C_n^2\cdot[ch(n-4)x-sh(n-4)x]+\Lambda.$$

So,

$$2^{n+1}\cdot ch^nx=(\alpha^n+\beta^n)+C_n^1\cdot(\alpha^{n-2}+\beta^{n-2})+C_n^2\cdot(\alpha^{n-4}+\beta^{n-4})+\Lambda$$
$$=2[chnx+C_n^1\cdot ch(n-2)x+C_n^2\cdot ch(n-4)x+\Lambda],$$

since, for every x∈**R** and every k∈$\mathbf{N}^*$,

$\alpha^k+\beta^k=2\cdot chkx$.

4) Let be x∈**R**:

$\alpha=chx+shx$ and $\beta=chx-shx$.

Then, according to the equalities (4.42‴) and (4.42$^{(iv)}$),

$$2^n\cdot sh^n x=\left(\alpha-\frac{1}{\alpha}\right)^n$$
$$=\alpha^n-C_n^1\cdot\alpha^{n-2}+C_n^2\cdot\alpha^{n-4}-\Lambda$$
$$=[chnx-shnx]-C_n^1\cdot[ch(n-2)x-sh(n-2)x]+C_n^2\cdot[ch(n-4)x-sh(n-4)x]-\Lambda$$

and:

$$2^n\cdot sh^n x=\left(\beta-\frac{1}{\beta}\right)^n$$
$$=\beta^n-C_n^1\cdot\beta^{n-2}+C_n^2\cdot\beta^{n-4}-\Lambda$$
$$=[chnx-shnx]-C_n^1\cdot[ch(n-2)x-sh(n-2)x]+C_n^2\cdot[ch(n-4)x-sh(n-4)x]-\Lambda\ .$$

So,

$$2^n\cdot[1-(-1)^n]\cdot sh^n x=(\alpha^n-\beta^n)-C_n^1\cdot(\alpha^{n-2}-\beta^{n-2})+C_n^2\cdot(\alpha^{n-4}-\beta^{n-4})-\Lambda$$
$$=2[shnx-C_n^1\cdot sh(n-2)x+C_n^2\cdot sh(n-4)x-\Lambda\ ],$$

since, for every $x\in\mathbf{R}$ and every $k\in\mathbf{N}^*$,

$$\alpha^k-\beta^k=2\cdot shkx.$$

5) We develop according to Newton's binomial the expressions $(chx+shx)^n$ and $(chx-shx)^n$ and we use the equalities $(4.42''')$ and $(4.42^{(iv)})$, by addition.

6) We develop according to Newton's binomial the expressions $(chx+shx)^n$ and $(chx-shx)^n$ and we use the equalities $(4.42''')$ and $(4.42^{(iv)})$, by subtraction.

It is observed that equality $(4.42^{(v)})$ is a generalization of equality (4.42), and equality $(4.42^{(vi)})$ is a generalization of equality (4.41), but expressed differently.

Returning now question before, I think the answer is closer to the truth: "*Do not know*".

In conclusion, we believe that we have achieved, for the attentive and interested reader of these issues, a fairly comprehensive picture of the properties of these functions. Now, teachers can apply these properties in various mathematical sub-branches, but also outside of Mathematics, or they can create an optional course such as: *Trigonometric functions versus hyperbolic functions* - in which to achieve a parallel approach to these two categories of functions.

Of course, we accept any pertinent critical remark or supplement to the list of properties of hyperbolic functions, from pupils, students or teachers.

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